The comparsion principle for viscosity solutions of fully nonlinear subelliptic equations in Carnot groups

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Abstract. *For any Carnot group* $\mathbf{G}$ *and a bounded domain* $\mathbf{\Omega} \subset \mathbf{G}$*, we prove that viscosity solutions in* $C(\bar{\Omega})$ *of the fully nonlinear subelliptic equation* $F(u, \nabla_h u, \nabla_h^2 u) = 0$ *are unique when* $F \in C(R \times R^m \times \mathcal{S}(m))$ *satisfies (i)* F *is degenerate subelliptic and decreasing in* u *or (ii)* F *is uniformly subelliptic and nonincreasing in* u*. This extends Jensen's uniqueness theorem from the Euclidean space to the sub-Riemannian setting of the Carnot group.*

§1. Introduction

The notion of viscosity solutions of fully nonlinear 2nd order degenerate elliptic equation:

$$F(x, u(x), \nabla u(x), \nabla^2 u(x)) = 0, \text{ in } R^n, \tag{1.1}$$

was developed by Crandall-Lions [CL] and Evans [E1,2] in 1980's. This idea, together with Jensen's celebrated uniqueness theorem [J1], provides a very satisfactory theory on existence, uniqueness, and compactness theorem of weak solutions of (1.1). The theory of viscosity solutions has been very powerful in many applications, and we refer to the user's guide [CIL] by Crandall-Ishii-Lions for many such applications.

In recent years there has been an explosion of interest in the study of analysis on sub-Riemannian, or Carnot-Carathédory spaces. The corresponding developments in the theory of partial differential equations of subelliptic type have prompted people to consider fully nonliear equations in Carnot groups. For examples, motivated by the very important work of Jensen [J2] on absolute minimizing Lipschitz extensions (or ALMEs, a notion first introduced by Aronsson [A]) and viscosity solutions to the ∞-laplacian equation in the Euclidean space, Bieske [B], Bieske-Capogna [BC], and Wang [W1] have studied absolute minimizing horizontal Lipschitz extensions and viscosity solutions to the ∞-sublaplacian equation on Carnot groups. In particular, the notion of viscosity solutions has been extended to fully nonlinear subelliptic equation (see [B]) and the uniqueness of viscosity solution of ∞-sublaplacian eqaution on any Carnot group was established by Wang [W1]. It is well-known (cf. the monographs [CC] by Caffarelli-Cabré and [G] by Gutierrez) that both convexity and the Monge-Ampére equation:

$$\det(\nabla^2 u) = f, \text{ in } R^n \tag{1.2}$$

have played crucial roles in the theory of fully nonlinear elliptic equation. Inspired by this, Lu-Manfredi-Stroffolini [LMS] and Danielli-Garofalo-Nhieu [DGN] have introduced and studied various notions of convexity, such as v-convexity and h-convexity, on Carnot groups (see also [BR], [W2], [JM] for some further related results). Moreover, Garofalo-Tournier [GN] and Gutierrez-Montanari [GM] have initiated the study of Monge-Ampére measures and maximum principle of convex functions on Heisenberg groups.

In this paper, we are interested in the comparison principle for viscosity solutions to 2nd order subelliptic equation which is either uniformly subelliptic, nonincreasing or degenerate subelliptic, decreasing in the sub-Riemannian setting of the Carnot group. In this aspect, we are able to extend Jensen's uniqueness theorem from the Euclean space to any Carnot group.

In order to describe our result, we first recall the basic properties of Carnot groups. A simply connected Lie group $\mathbf{G}$ is called a Carnot group of step $r \geq 1$, if its Lie algebra g admits a vector space decomposition in r layers $g = V_1 + V_2 + \cdots + V_r$ such that (i) g is stratified, i.e., $[V_1, V_j] = V_{j+1}, j = 1, \cdots, r-1$, and (ii) g is r-nilpotent, i.e. $[V_j, V_r] = 0, j = 1, \cdots, r$. We call V_1 the *horizontal* layer and $V_j, j = 2, \cdots, r$ the *vertical* layers. We choose an inner product $\langle \cdot, \cdot \rangle$ on g such that $V_j' s$ are mutually orthogonal for $1 \leq j \leq r$. Let $\{X_{j,1}, \cdots, X_{j,m_j}\}$ denote a fixed orthonormal basis of V_j for $1 \leq j \leq r$, where $m_j = \dim(V_j)$ is the dimension of V_j. From now on, we also denote $m = \dim(V_1)$ as the dimension of the horizontal layer and set $X_i = X_{1,i}$ for $1 \leq i \leq m$. It is well-known (see [FS]) that the exponential map $exp : g \equiv R^n \to G$ is a global diffeomorphism and yields an exponential coordinate system on $\mathbf{G}$, with $n = \sum_{i=i}^r m_i$ the topological dimension of $\mathbf{G}$. More precisely, any $p \in \mathbf{G}$ has a coordinate $((p_1, \cdots, p_m), (p_{2,1}, \cdots, p_{2,m_2}), \cdots, (p_{r,1}, \cdots, p_{r,m_r}))$ such that

$$p = exp(\xi_1(p) + \cdots \xi_r(p)), \text{ with } \xi_1(p) = \sum_{l=1}^m p_l X_l, \ \xi_i(p) = \sum_{j=1}^{m_i} p_{i,j} X_{i,j}, 2 \leq i \leq r.$$

The exponential map can induce a homogeneous pseudo-norm $N_{\mathbf{G}}$ on $\mathbf{G}$ in the following way (see [FS]).

$$N_{\mathbf{G}}(p) := (\sum_{i=1}^r |\xi_i(p)|^{\frac{2r!}{i}})^{\frac{1}{2r!}}, \text{ if } p = exp(\xi_1(p) + \cdots \xi_r(p)), \tag{1.3}$$

where $|\xi_1(p)| = (\sum_{l=1}^m p_l^2)^{\frac{1}{2}}$, and $|\xi_i(p)| = (\sum_{j=1}^{m_i} p_{i,j}^2)^{\frac{1}{2}} (2 \leq i \leq r)$. Moreover, $N_{\mathbf{G}}$ yields a pseudo-distance on $\mathbf{G}$ as follows.

$$d_{\mathbf{G}}(p, q) := N_{\mathbf{G}}(p^{-1} \cdot q), \ \forall p, \ q \in \mathbf{G}, \tag{1.4}$$

where $\cdot$ is the group multiplication of $\mathbf{G}$ and p^{-1} is the inverse of p. It is easy to see that $d_{\mathbf{G}}$ satisfies the invariance property

$$d_{\mathbf{G}}(z \cdot x, z \cdot y) = d_{\mathbf{G}}(x, y), \quad \forall x, \ y, \ z \in G, \tag{1.5}$$

and is of homogeneous of degree one, i.e.

$$d_{\mathbf{G}}(\delta_\lambda(p),\delta_\lambda(q))=\lambda d_{\mathbf{G}}(p,q),\ \forall\lambda>0,\ \forall p,\ q\in\mathbf{G} \tag{1.6}$$

where $\delta_\lambda(p)=\lambda\xi_1(p)+\sum_{i=2}^r\lambda^i\xi_i(p)$ is the non-isotropic dilations on $\mathbf{G}$.

Throughout this paper, we fix some notations. For $l\geq 1$, denote $\mathcal{S}(l)$ as the set of $l\times l$ symmetric matrices. For $M,N\in\mathcal{S}(m)$, we say $M\geq N$ if $(M-N)\in\mathcal{S}(m)$ is a positive semidefinite matrix, and let trace(M) denote the trace of M for $M\in\mathcal{S}(m)$. For $u:\mathbf{G}\to R$, let $\nabla u,\nabla^2u$ denote the Euclidean gradient, hessian of u respectively, and $\nabla_h u:=(X_1u,\cdots,X_mu)$, $\nabla_h^2u:=(\frac{X_iX_j+X_jX_i}{2}u)_{1\leq i,j\leq m}$ denote the horizontal gradient, horizontal hessian of u respectively. For a given domain $\Omega\subset\mathbf{G}$, denote $C(\Omega)$ as the set of continuous functions on Ω, $C^2(\Omega)=\{u\in C(\Omega):\nabla u,\nabla^2u\in C(\Omega)\}$, and $\Gamma^2(\Omega)=\{u\in C(\Omega):\nabla_hu,\nabla_h^2u\in C(\Omega)\}$. A fully nonlinear partial horizontal-differential operator $\mathcal{F}[\cdot]$ on Ω is defined by

$$\mathcal{F}[\phi](x)=F(\phi(x),\nabla_h\phi(x),\nabla_h^2\phi(x)),\ \forall x\in\Omega,\ \forall\phi\in\Gamma^2(\Omega), \tag{1.7}$$

where $F\in C(R\times R^m\times\mathcal{S}(m))$. We now give the definition of subellipticity and nondecreasing property of $\mathcal{F}$.

Definition 1.1. The operator $\mathcal{F}[\cdot]$ is degenerate subelliptic if

$$F(r,p,M)\leq F(r,p,N),\ \text{for all } M,N\in\mathcal{S}(m) \text{ with } M\leq N \text{ and } (r,p)\in R\times R^m. \tag{1.8}$$

The operator $\mathcal{F}[\cdot]$ is uniformly subelliptic if there exist constants $\alpha_1,\alpha_2>0$ such that

$$F(r,p,M)-F(r,q,N)\geq\alpha_1\text{trace}(M-N)-\alpha_2|p-q| \tag{1.9}$$

for all $M,N\in\mathcal{S}(m)$ with $M\geq N$ and $(r,p,q)\in R\times R^m\times R^m$.

Definition 1.2. The operator $\mathcal{F}[\cdot]$ is nonincreasing if

$$F(r,p,M)\leq F(s,p,M),\quad \text{for all } r\geq s,\ \text{and } (p,M)\in R^m\times\mathcal{S}(m). \tag{1.10}$$

The operator $\mathcal{F}[\cdot]$ is decreasing if there is a constant $\alpha_3>0$ such that

$$F(r,p,M)-F(s,p,M)\leq\alpha_3(s-t) \text{ for all } r\geq s,\ \text{and } (p,M)\in R^m\times\mathcal{S}(m). \tag{1.11}$$

We shall now recall the definition of viscosity solution of fully nonlinear degenerate subelliptic equation (1.7), which was introduced by Crandall-Lions (see [CL] and the user's guides [CIL]) for fully nonlinear elliptic equasions.

Definition 1.3. Assume that $\mathcal{F}$ is a degenerate subelliptic operator. $w \in C(\Omega)$ is a viscosity subsolution of (1.7) if for any $(x_0, \phi) \in \Omega \times C^2(\Omega)$ such that

$$0 = \phi(x_0) - w(x_0) \geq \phi(x) - w(x), \forall x \in \Omega$$

we have

$$F(\phi(x_0), \nabla_h \phi(x_0), \nabla_h^2 \phi(x_0)) \geq 0. \tag{1.12}$$

$w \in C(\Omega)$ is a viscosity supersolution of (1.7) if $-w$ is a viscosity subsolution of (1.7). $w \in C(\Omega)$ is a viscosity solution of (1.7) if it is both a viscosity subsolution and a viscosity supersolution.

Remark 1.4. (i) It is well-known that there is an equivalent formulation of viscosity solution of (1.7) using elliptic jets (see [CIL] or [J1]§2). (ii) From the sub-Riemannian point of views, it also seems natural to define an intrinsic version of viscosity solution of (1.7) by allowing the test functions $\phi \in \Gamma^2(\Omega)$ in Definition 1.3. However, since $C^2(\Omega) \subset \Gamma^2(\Omega)$, the intrinsic version of viscosity solution of (1.7) is stronger than the version given by Definition 1.3. (iii) This intrinsic version of viscosity solution of (1.7) has been previously formulated by Bieske [B] (see also Manfredi [M]) in connections with ∞-sublaplacian equations on Heisenberg groups, where the notion of subelliptic jets was also introduced.

Now we are ready to state our comparison theorem.

Theorem A. *Let* $\mathbf{G}$ *be a Carnot group and* $\Omega \subset \mathbf{G}$ *be a bounded domain. Suppose that* $u \in C(\bar{\Omega})$ *is a viscosity subsolution of (1.7) and* $v \in C(\bar{\Omega})$ *is a viscosity supersolution of (1.7). If* F *satisfies either*

(i) $\mathcal{F}[\cdot]$ *is degenerate subelliptic and decreasing,*

or

(ii) $\mathcal{F}[\cdot]$ *is uniformly subelliptic and nonincreasing,*

then

$$\sup_{\Omega}(u - v)^+ \leq \sup_{\partial\Omega}(u - v)^+. \tag{1.13}$$

We would like to remark that the operator $\bar{\mathcal{F}}[\cdot]$ induced by the degenerate subelliptic operator $\mathcal{F}[\cdot]$:

$$\bar{\mathcal{F}}[w] = \bar{F}(x, w, \nabla w, \nabla^2 w) := F(w, \nabla_h w, \nabla_h^2 w) \tag{1.14}$$

may not be degenerate elliptic (i.e $\bar{F}$ may not be monotone in its third variable, see [CIL]), and may be dependent of the spatial variable x in an essential way so that the uniqueness theorems by Jensen [J1] and Ishii [I] on viscosity solutions to 2nd order elliptic equations are not applicable here. Therefore theorem A not only provides a comparison principle in

the subelliptic setting of the Carnot group but also makes the comparison principle of [J1] (see also [I] or [CIL]) available for a considerably larger class of equations in the Euclidean setting.

We believe that theorem A shall play an important role in the existence of viscosity of (1.7) by the Perron's method (see [I]) and plan to study it in a future article. We would like to mention that Manfredi [M] proved, among other things, theorem A for any uniformly elliptic, linear subelliptic opertor $F(w) = \sum_{i,j=1}^{m} a_{ij}(x) X_i X_j w$, with $(a_{ij}(x)) \in C(\mathbf{G}, \mathcal{S}(m))$ a uniformly elliptic matrix.

A direct consequence of Theorem A is the uniqueness theorem of viscosity solutions of (1.7).

Corollary B. *Under the same assumptions as Theorem A. There exists at most one viscosity solution $u \in C(\bar{\Omega})$ of (1.7).*

The basic point to prove theorem A is that we can always compare between a *classcial* subsolution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and a *classical strict* supersolution $v \in C^2(\Omega) \cap C(\bar{\Omega})$ of (1.7), under the assumption that $\mathcal{F}[\cdot]$ is degenerate subelliptic.

In order to extend this idea to viscosity sub (or super) solutions of (1.7), we first establish, in Lemma 2.1 below, that under the same conditions of theorem A, any viscosity supersolution $v \in C(\bar{\Omega})$ can be perturbed into a viscosity *strict* supersolution of (1.7). We would like to point out that, even in the Eucliean setting of R^n, Lemma 2.1 seems to be new and can be used to simplify the proof of Jensen [J1]. Moreover, it seems necessary in the subelliptic setting, since the counterpart of [J1] Lemma 3.20 is to estimate $\frac{\mathrm{trace}(\nabla_h^2 w)^-}{|\nabla_h w|}$ from below for a semiconvex function w on $\mathbf{G}$ and may not be available.

The second ingredient is to approximate viscosity sub (or super) solutions by semiconvex (or semiconcave) sub (or super) solutions of (1.7). This idea was first introduced by Jensen in his very important paper [J1] on uniqueness of Lipschitz continuous viscosity solutions to 2nd order elliptic equations, and Jensen's original approximation scheme was further simplified by the sup/inf convolution construction by Jensen-Lions-Souganidis [JLS] in the Euclidean setting. In the subelliptic setting of the Carnot group, we succeeded, in an earlier paper [W1] where we proved the uniqueness of viscosity solution to the subelliptic ∞-laplacian equation on any Carnot group $\mathbf{G}$, in extending the sup/inf convolution construction of [JLS] by employing the smooth gauge pseudo-norm function $d_{\mathbf{G}}$ and get the desired approximations. For the reader's convenience, we review the sup/inf convolution construction of [W1] in §3 below. We would like to point out that we have used in a very crucial way that (1.7) is invariant under the group multiplication from left, i.e. if $u \in C(\mathbf{G})$ is a viscosity solution to (1.7) then $u_a(x) = u(a \cdot x),\ x \in \mathbf{G}$, is also a viscosity solution to (1.7) for any $a \in \mathbf{G}$. Once we have semiconvex (or semiconcave) sub (or super) solutions to (1.7), we can apply both the regularity properties (see Evans-Gariepy [EG])

and Jensen's maximal principle for semiconvex functions (see [J1]) in our setting.

The paper is written as follows. In §2, we show that any viscosity supersolution of (1.7) given by theorem A can be perturbed into a *strict* supersolution of (1.7). In §3, we recall the sup/inf convolution construction on a Carnot group $\mathbf{G}$, which was carried out in an earlier paper [W1]. In §4, we give a proof of theorem A.

§2 Viscosity strict supersolutions

In this section, we show that any viscosity supersolution given by theorem A can be perturbed into a viscosity strict supersolution by a suitable small perturbation. More precisely, we have

Lemma 2.1. *Suppose that $F \in C(R \times R^m \times \mathcal{S}(m))$ and $v \in C(\Omega)$ is a viscosity supersolution to*

$$\mathcal{F}[w] := F(w, \nabla_h w, \nabla_h^2 w) = 0, \ \text{in} \ \ \Omega, \tag{2.1}$$

under either (i) $\mathcal{F}$ is degenerate subelliptic and decreasing or (ii) $\mathcal{F}$ is uniformly subelliptic, nonincreasing. Then for any $\delta \in (0, \delta_0)$ there are $c_\delta > 0$ and $v^\delta \in C(\Omega)$ so that

$$v(x) \le v^\delta(x) \le v(x) + \delta \tag{2.2}$$

and v^δ is a viscosity supersolution to

$$F(w, \nabla_h w, \nabla_h^2 w) + c_\delta = 0, \ \text{in} \ \ \Omega. \tag{2.3}$$

Proof. We first recall that for $x \in \mathbf{G}$ if $(x_1, \cdots, x_m)$ denotes the horizontal component of its coordinate and $((x_{2,1}, \cdots, x_{2,m_2}), \cdots, (x_{r,1}, \cdots, x_{r,m_r}))$ denotes the vertical component of its coordinate, then the horizontal vector fields $X_l, 1 \le l \le m$, can be expressed as (see [FS])

$$X_l = \frac{\partial}{\partial x_l} + \sum_{i=2}^{r} \sum_{j=1}^{m_i} a_{ij}(x) \frac{\partial}{\partial x_{i,j}}, \ 1 \le l \le m, \tag{2.4}$$

where $\{a_{ij}\}$ are smooth on $\mathbf{G}$ for $2 \le i \le r, 1 \le j \le m_i$. For $k > 0$, denote $c_1 = \inf_{x \in \Omega} x_1 \in R$ and define

$$\alpha_k(x) = 1 - \frac{1}{k} e^{-k(x_1 + 1 - c_1)}, \ \forall x \in \Omega.$$

Then (2.4) implies that, for any $x \in \Omega$, we have

$$X_1 \alpha_k(x) = e^{-k(x_1 + 1 - c_1)}, \ X_2 \alpha_k(x) = \cdots = X_m \alpha(x) = 0, \tag{2.5}$$

$$X_{ij}\alpha_k(x) = -ke^{-k(x_1+1-c_1)}, \text{ if } i=j=1, \qquad (2.6)$$
$$= 0, \qquad \text{otherwise.}$$

For any $\delta > 0$ and $k \geq 2$ to be chosen later, we consider $v^\delta(x) := v(x) + \delta\alpha_k(x) : \Omega \to R$. Since $0 \leq \alpha_k \leq 1$, it is easy to see that v^δ satisfies (2.2). We want to show that v^δ is also a viscosity supersolution to eqn. (2.4). To do this, let $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ touch v^δ from below at $x = x_0$, i.e.

$$0 = (v + \delta\alpha_k - \phi)(x_0) \geq (v + \delta\alpha_k - \phi)(x), \ \forall x \in \Omega.$$

This implies that $\phi - \delta\alpha_k \in C^2(\Omega)$ touches v from below at x_0. Since v is a viscosity supersolution to eqn.(2.1), we have

$$F(\phi - \delta\alpha_k, \nabla_h\phi - \delta\nabla_h\alpha_k, \nabla_h^2\phi - \delta\nabla_h^2\alpha_k)(x_0) \leq 0. \qquad (2.7)$$

Now we need to show that there exists a $k_0 > 0$ such that for any $k \geq k_0$ (2.3) is true. We proceed it as follows.

Case 1. $\mathcal{F}$ is degenerate subelliptic and decreasing :

It follows from (2.6) that $\nabla_h^2(-\alpha_k)$ is positive semidefinite. Therefore, the degenerate subellipticity (1.8) and decreasing property (1.11) of $\mathcal{F}$ imply

$$\begin{aligned}
&F(\phi - \delta\alpha_k, \nabla_h\phi - \delta\nabla_h\alpha_k, \nabla_h^2\phi - \delta\nabla_h^2\alpha_k)(x) \\
&\geq F(\phi - \delta\alpha_k, \nabla_h\phi - \delta\nabla_h\alpha_k, \nabla_h^2\phi)(x) \\
&\geq F(\phi, \nabla_h\phi - \delta\nabla_h\alpha_k, \nabla_h^2\phi)(x) + \alpha_3\delta\alpha_k(x) \\
&\geq F(\phi, \nabla_h\phi, \nabla_h^2\phi)(x) + \alpha_3\delta\alpha_k(x) - \omega_2(\delta|\nabla_h\alpha_k|(x)), \qquad (2.8)
\end{aligned}$$

where ω_2 is the modular of continuity of F with respect to it second variable. Since $\alpha_k(x) \geq \frac{1}{2}$ and $|\nabla_h\alpha_k|(x) \leq \frac{1}{2k}$, (2.8) implies

$$F(\phi, \nabla_h\phi, \nabla_h^2\phi)(x_0) \leq \omega_2(\delta k^{-1}) - \frac{k_0\delta}{2} \equiv c_\delta < 0, \qquad (2.9)$$

if we choose k so large that $\omega_2(\delta k^{-1}) \leq \frac{k_0\delta}{4}$. This verifies (2.4) under the condition (i) of Lemma 2.1.

Case 2. $\mathcal{F}$ is uniformly subelliptic, nonincreasing :

Since $\nabla_h^2(-\alpha_k)$ is positive semidefinite, the uniform ellipticity (1.9) and nonincreasing property (1.10) of $\mathcal{F}$ imply

$$\begin{aligned}
&F(\phi - \delta\alpha_k, \nabla_h\phi - \delta\nabla_h\alpha_k, \nabla_h^2\phi - \delta\nabla_h^2\alpha_k)(x) \\
&\geq F(\phi, \nabla_h\phi - \delta\nabla_h\alpha_k, \nabla_h^2\phi)(x) + \alpha_1\text{trace}(\nabla_h^2(-\alpha_k))(x) \\
&\geq F(\phi, \nabla_h\phi - \delta\nabla_h\nabla_k, \nabla_h^2\phi)(x) + \alpha_1 ke^{-k(x_1+1-c_1)} \\
&\geq F(\phi, D_h\phi, D_h^2\phi)(x) + \alpha_1 ke^{-k(x_1+1-c_1)} - \alpha_2\delta|\nabla_h\alpha_k|(x) \\
&\geq F(\phi, D_h\phi, D_h^2\phi)(x) + e^{-k(x_1+1-c_1)}(\alpha_1 k - \alpha_2\delta). \qquad (2.10)
\end{aligned}$$

Therefore if we choose $k \geq \frac{2\alpha_2\delta}{k_1}$, then we have

$$F(\phi, \nabla_h\phi, \nabla_h^2\phi)(x_0) \leq -\frac{\alpha_2\delta}{2}e^{-k(x_1+1-c_1)}, \tag{2.11}$$

this, combined with $\inf_{x\in\Omega} e^{-k(x_1+1-c_1)} = 2c_2(k) > 0$, implies that (2.4) holds with $c_\delta = c_2(k)\alpha_2\delta > 0$. Therefore, the proof of Lemma 2.1 is complete. ■

§3. The construction of sup/inf convolutions on $\mathbf{G}$

For the convenience of readers, we recall the construction of sup/inf convolution on any Carnot group $\mathbf{G}$, which was carried out earlier by Wang [W1]. The key observation is that the equation (1.7) is invariant under group multiplication from left on $\mathbf{G}$. We would like to point out that this construction is an extension of that by Jensen-Lions-Souganidis [JLS] in the Euclidean space.

Let $\Omega \subset \mathbf{G}$ be a bounded domain and $d_{\mathbf{G}}(\cdot,\cdot)$ be the smooth gauge distance defined by (1.3). For any $\epsilon > 0$, define

$$\Omega_\epsilon = \{x \in \Omega : \inf_{y\in\mathbf{G}\setminus\Omega} d_{\mathbf{G}}(x^{-1}, y^{-1}) \geq \epsilon\}.$$

Definition 3.1. For any $u \in C(\bar{\Omega})$ and $\epsilon > 0$, the sup involution u_ϵ of u is defined by

$$u^\epsilon(x) = \sup_{y\in\bar{\Omega}}(u(y) - \frac{1}{2\epsilon}d_{\mathbf{G}}(x^{-1}, y^{-1})^{2r!}),\ \forall x \in \Omega. \tag{3.1}$$

Similarly, the inf involution v_ϵ of $v \in C(\bar{\Omega})$ is defined by

$$v_\epsilon(x) = \inf_{y\in\bar{\Omega}}(v(y) + \frac{1}{2\epsilon}d_{\mathbf{G}}(x^{-1}, y^{-1})^{2r!}),\ \forall x \in \Omega. \tag{3.2}$$

For $p \in \mathbf{G}$, let $\|p\|_E := (\sum_{i=1}^r |\xi_i(p)|^2)^{\frac{1}{2}}$ be the euclidean norm of p. We recall

Definition 3.2. A function $u \in C(\bar{\Omega})$ is called semiconvex, if there is a constant $C > 0$ such that $u(p) + C\|p\|_E^2$ is convex in the Euclidean sense; and u is called semiconcave if $-u$ is semiconvex. Note that, for $u \in C^2(\Omega)$, if $\nabla^2 u(p) + C$ is positive semidefinite for any $p \in \Omega$, then u is semiconvex.

Now we have

Proposition 3.3. *For $u, v \in C(\bar{\Omega})$, denote $R_0 = \max\{\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}\}$. Then, for any $\epsilon > 0$, $u^\epsilon, v_\epsilon \in W^{1,\infty}_{CC}(\Omega)$ satisfy*

(1) u^ϵ is semiconvex and v_ϵ is semiconcave.

(2) u^ϵ is monotonically nondecreasing w.r.t. ϵ and converges uniformly to u on Ω; and v_ϵ is monotonically nonincreasing w.r.t. ϵ and converges uniformly to v on Ω.

(3) if u (or v respectively) is a viscosity subsolution (or supersolution respectively) to a degenerate subelliptic equation:

$$F(u, \nabla_h u, \nabla_h^2 u) = 0 \quad in \quad \Omega, \tag{3.3}$$

where $F \in C(R \times R^m \times S(m))$. Then u^ϵ (or v_ϵ) is a viscosity subsolution (or supersolution respectively) to eqn. (3.3) in $\Omega_{2R_0\epsilon}$.

Proof. Since the proof of v_ϵ can be done by the same wasy as that of u^ϵ, it suffices to consider u^ϵ. For $\Omega \subset \mathbf{G}$ is bounded, the formula (1.3) of $d_{\mathbf{G}}$ implies

$$C(\Omega, d_{\mathbf{G}}) \equiv \|\nabla_x^2 (d_{\mathbf{G}}(x^{-1}, y^{-1})^{2r!})\|_{L^\infty(\Omega \times \Omega)} < \infty.$$

Therefore, for any $y \in \bar{\Omega}$, the full hessian of

$$\tilde{u^\epsilon}(x, y) := u(y) - \frac{1}{2\epsilon} d_{\mathbf{G}}(x^{-1}, y^{-1})^{2r!} + \frac{C(\Omega, d_{\mathbf{G}})}{2\epsilon} \|x\|_E^2, \ \forall x \in \Omega,$$

is positive semidefinite so that $\tilde{u^\epsilon}$ is convex. Note that the superum for a family of convex functions is still convex, this implies that

$$u_\epsilon(x) + \frac{C(\Omega, d_{\mathbf{G}})}{2\epsilon} \|x\|_E^2 = \sup_{y \in \bar{\Omega}} \tilde{u^\epsilon}(x, y), \ \forall x \in \Omega$$

is convex so that u_ϵ is semiconvex. It is well-known that semiconvex functions are Lipschitz continuous with respect to the euclidean metric (cf. Evans-Gariepy [EG]). Therefore u^ϵ is Lipschitz continuous in Ω with respect to $d_{\mathbf{G}}$. This gives (1).

For any $\epsilon_1 < \epsilon_2$, it is easy to see that $u^{\epsilon_1}(x) \le u^{\epsilon_2}(x)$ so that $\{u^\epsilon\}$ is monotonically nondecreasing with respect to ϵ. Observe that for any $x \in \Omega$ there exists a $x_\epsilon \in \bar{\Omega}$ such that

$$u(x) \le u^\epsilon(x) = u(x_\epsilon) - \frac{1}{2\epsilon} d_{\mathbf{G}}(x^{-1}, x_\epsilon^{-1}) \le R_0. \tag{3.4}$$

This implies

$$u(x_\epsilon) - u^\epsilon(x) = \frac{1}{2\epsilon} d_{\mathbf{G}}(x^{-1}, x_\epsilon^{-1}) \le u(x_\epsilon) - u(x) = \omega_u(\|x_\epsilon - x\|_E), \ \forall x \in \Omega, \tag{3.5}$$

where ω_u denotes the modular of continuity of u. On the other hand, the monotonicity of u^ϵ with respect to ϵ implies

$$u_{\frac{\epsilon}{2}}(x) \ge u(x_\epsilon) - \frac{1}{\epsilon} d(x^{-1}, x_\epsilon^{-1})^{2r!} = u_\epsilon(x) - \frac{1}{2\epsilon} d(x^{-1}, x_\epsilon^{-1})^{2r!}$$

so that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}d(x^{-1},x_\epsilon^{-1})^{2r!}=0,\ \forall x\in\Omega. \tag{3.6}$$

This implies that $\lim_{\epsilon\to 0} x_\epsilon = x$ and $\lim_{\epsilon\to 0} u^\epsilon(x) = u(x)$ for any $x \in \Omega$. Moreover, (3.5) implies

$$d_{\mathbf{G}}(x^{-1},x_\epsilon^{-1})\le 2\epsilon\omega_u(\|x_\epsilon - x\|_E)\le 2R_0\epsilon \tag{3.7}$$

so that $\|x_\epsilon - x\|_E \le C(R_0\epsilon)^{\frac{1}{r}}$, where r is the step of $\mathbf{G}$. This, combined with (3.5) again, implies

$$\max_{x\in\Omega}|u^\epsilon(x)-u(x)|\le\omega_u(C(R_0\epsilon)^{\frac{1}{r}})\to 0,\quad \text{as } \epsilon\to 0$$

so that u^ϵ converges to u uniformly. Therefore (2) is proved.

For (3), we first observe that (3.7) implies that for $x^0 \in \Omega_{2R_0\epsilon}$, $u^\epsilon(x^0)$ is attained by a $x_\epsilon^0 \in \Omega$. Now we let $\phi \in C^2(\Omega)$ be such that

$$u^\epsilon(x^0)-\phi(x^0)\ge u^\epsilon(x)-\phi(x),\quad \forall x\in\Omega_{2R_0\epsilon}.$$

Then we have, for any $x, y \in \Omega_{2R_0\epsilon}$,

$$u(x_\epsilon^0)-\frac{1}{2\epsilon}d_{\mathbf{G}}((x^0)^{-1},(x_\epsilon^0)^{-1})^{2r!}-\phi(x^0)\ge u(y)-\frac{1}{2\epsilon}d_{\mathbf{G}}(x^{-1},y^{-1})^{2r!}-\phi(x). \tag{3.8}$$

For y near x_ϵ^0, since $x = x^0\cdot(x_\epsilon^0)^{-1}\cdot y \in \Omega_{2R_0\epsilon}$, we can substitue x into (3.8) to get

$$u(x_\epsilon^0)-\phi(x^0)\ge u(y)-\phi(x^0\cdot(x_\epsilon^0)^{-1}\cdot y).$$

Set $\bar\phi(y) = \phi(x^0\cdot(x_\epsilon^0)^{-1}\cdot y)$ for $y \in \Omega_{2R_0\epsilon}$ close to y_0. Then $\bar\phi$ touches u from above at $y = x_\epsilon^0$ so that u being a viscosity subsolution of eqn. (3.3) implies

$$F(u(x_\epsilon^0),\nabla_h\bar\phi(x_\epsilon^0),\nabla_h^2\bar\phi(x_\epsilon^0))\le 0. \tag{3.9}$$

Note that the left-invariance of X_i, we know

$$\nabla_h\bar\phi(y)=\nabla_h\phi(x^0\cdot(x_\epsilon^0)^{-1}\cdot y),\quad \nabla_h^2\bar\phi(y)=\nabla_h^2\phi(x^0\cdot(x_\epsilon^0)^{-1}\cdot y).$$

Hence we have

$$F(u(x_\epsilon^0),\nabla_h\phi(x_0),\nabla_h^2\phi(x_0))\le 0. \tag{3.10}$$

Taking ϵ into zero, (3.10) implies that u^ϵ is a viscosity subsolution of eqn.(3.3) on $\Omega_{2R_0\epsilon}$. The proof is complete. ■

§4. Proof of Theorem A

This section is devoted to the proof of the comparison Theorem. The idea is to prove the comparison property between the strict supersolution obtained by Lemma 2.1 and the subsolution by comparing their sup/inf convolutions. The almost everywhere twice differentiablity ([EG]) and Jensen's maximum principle ([J1,2]) for semiconvex functions play very important roles in this aspect.

Through this section, we express the horizontal vector fields X_i by the formula (2.4).

Proof of Theorem A.

Suppose that (1.13) were fasle. Then

$$\delta_0 = \sup_{\bar{\Omega}} (u-v)^+ - \sup_{\partial\Omega} (u-v)^+ > 0. \tag{4.1}$$

Denote $c^+ = \sup_{\partial\Omega}(u-v)^+ \geq 0$. Note that $v + c^+$ is also a viscosity supersolution to eqn.(?), and (4.1) implies

$$\delta_0 = \sup_{\bar{\Omega}} (u-(v+c^+))^+ - \sup_{\partial\Omega} (u-(v+c^+))^+ > 0.$$

Hence we may assume $c^+ = 0$ (i.e. $u(x) \leq v(x)$ for any $x \in \partial\Omega$) so that (4.1) implies $\delta_0 = \sup_{\Omega}(u-v) > 0$.

For any $\delta \in (0, \frac{\delta_0}{4})$, let $v^\delta \in C(\bar{\Omega})$ be the strict supersolution of eqn.(1.7) given by Lemma 2.1. In particular, v^δ is a viscosity supersoltution to

$$F(w, \nabla_h w, \nabla_h^2 w) + c_\delta = 0, \text{ in } \Omega. \tag{4.2}$$

For any $\epsilon \in (0, \delta)$, we now let u^ϵ (v_ϵ^δ, respectively) be the sup-convolution (inf-convolution, respectively) of u (v^δ respectively) given by Proposition 3.3. By considering a smaller domain, we may assume that u^ϵ is a viscosity subsolution of eqn.(1.7) and v_ϵ^δ is a viscosity supersolution of eqn.(4.2) in Ω, and

$$\sup_{\bar{\Omega}} (u^\epsilon - v_\epsilon^\delta) > 0 \geq \sup_{\partial\Omega} (u^\epsilon - v_\epsilon^\delta)$$

is achieved at a point $x_0 \in \Omega$. Since Proposition 3.3 implies that $u^\epsilon - v_\epsilon^\delta$ is semiconvex, we know (cf. [J2] page 67) that

$$\nabla u^\epsilon(x_0), \nabla v_\epsilon^\delta(x_0) \text{ both exist and are equal,} \tag{4.3}$$

$$u^\epsilon(x) - u^\epsilon(x_0) - \langle \nabla u^\epsilon(x_0), x - x_0 \rangle_E = O(\|x-x_0\|_E^2), \tag{4.4}$$

$$v_\epsilon^\delta(x) - v_\epsilon^\delta(x_0) - \langle \nabla v(x_0), x - x_0 \rangle_E = O(\|x-x_0\|_E^2), \tag{4.5}$$

where $\langle\cdot,\cdot\rangle_E$ denotes the Euclidean inner product on $\mathbf{G}$. Let $R_0 = \text{dist}_E(x_0, \partial\Omega) = \inf_{x\in\partial\Omega}\|x_0 - x\|_E > 0$ be the euclidean distance from x_0 to $\partial\Omega$ and $R_1 > 0$ be such that both (4.4) and (4.5) hold with $\|x - x_0\| < R_1$. Set $R_2 = \min\{R_0, R_1\} > 0$. For simplicity, we will denote u, v as u^ϵ, v_ϵ^δ respectively from now on. For any small $\rho > 0$, define the rescaled maps u^ρ, v^ρ in the euclidean ball $B^E_{R_2\rho^{-1}}$ by

$$u^\rho(x) = \frac{1}{\rho^2}(u(x_0 + \rho x) - u(x_0) - \rho\langle\nabla u(x_0), x\rangle_E),$$
$$v^\rho(x) = \frac{1}{\rho^2}(v(x_0 + \rho x) - v(x_0) - \rho\langle\nabla v(x_0), x\rangle_E),$$

where we have used the Euclidean addition and scalar multiplication. Then it is easy to see

$$(u^\rho - v^\rho)(0) > 0,\ (u^\rho - v^\rho)(0) \geq (u^\rho - v^\rho)(x),\ \forall x \in B^E_{R_2\rho^{-1}}. \tag{4.6}$$

It follows from (4.4) and (4.5) that, for any $R > 0$, there exists an $\rho_0 = \rho_0(R) > 0$ such that (i) $\{u^\rho\}_{\{0<\rho\leq\rho_0\}}$ are uniformly bounded, uniformly semiconvex, and uniformly Lipschitz continuous in B^E_R; and (ii) $\{v^\rho\}_{\{0<\rho\leq\rho_0\}}$ are uniformly bounded, uniformly semiconcave, and uniformly Lipschitz continuous in B^E_R. Therefore, by the Cauchy diagonal process, we may assume that there is $\rho_i \downarrow 0$ such that $u^{\rho_i} \to u^*$, $v^{\rho_i} \to v^*$ locally uniformly in R^n, where $n = \dim(\mathbf{G})$. In particular, (i) and (ii) imply that u^* is locally bounded, semiconvex in R^n, and v^* is locally bounded, semiconcave in R^n, and

$$(u^* - v^*)(0) > 0,\ (u^* - v^*)(0) \geq (u^* - v^*)(x), \quad \forall x \in R^n.$$

Now we have

Claim 4.1. *u^* satisfies, in the sense of viscosity,*

$$F(u(x_0), \nabla_h u(x_0), \{\sum_{k,l=1}^n (A^{ij}_{kl}\frac{\partial^2 u^*}{\partial x_k \partial x_l} + a_{ik}(x_0)\frac{\partial a_{jl}}{\partial x_k}(x_0)\frac{\partial u}{\partial x_l}(x_0)\}_{1\leq i,j\leq m}) \geq 0,\ \textit{in } R^n, \tag{4.7}$$

and v^* satisfies, in the sense of viscosity,

$$F(v(x_0), \nabla_h v(x_0), \{\sum_{k,l=1}^n (A^{ij}_{kl}\frac{\partial^2 v^*}{\partial x_k \partial x_l} + a_{ik}(x_0)\frac{\partial a_{jl}}{\partial x_k}(x_0)\frac{\partial v}{\partial x_l}(x_0)\}_{1\leq i,j\leq m}) + c_\delta \leq 0,\ \textit{in } R^n, \tag{4.8}$$

where $A^{ij}_{kl} = a_{ik}(x_0)a_{jl}(x_0)$, for $1 \leq k, l \leq n, 1 \leq i, j \leq m$, and $c_\delta > 0$.

Let's assume Claim 4.1 for the moment and proceed as follows. Since $u^* - v^*$ is semiconvex and achieves its maximum at $x = 0$, we can apply Jensen's maximum principle for semiconvex functions (see [J1] [J2]) to conclude that there exists $x_* \in R^n$ such

that $\nabla^2 u^*(x_*), \nabla^2 v^*(x_*)$ both exist and $\nabla^2(u^*-v^*)(x_*)$ is negative semidefinite. Denote $M_1, M_2 \in \mathcal{S}(m)$ by

$$M_1^{ij} = \sum_{k,l=1}^{n} \{A_{kl}^{ij} \frac{\partial^2 u^*}{\partial x_k \partial x_l}(x_*) + a_{ik}(x_0) \frac{\partial a_{jl}}{\partial x_k}(x_0) \frac{\partial u}{\partial x_l}(x_0)\},\ 1 \le i, j \le m,$$

and

$$M_2^{ij} = \sum_{k,l=1}^{n} \{A_{kl}^{ij} \frac{\partial^2 v^*}{\partial x_k \partial x_l}(x_*) + a_{ik}(x_0) \frac{\partial a_{jl}}{\partial x_k}(x_0) \frac{\partial v}{\partial x_l}(x_0)\},\ 1 \le i, j \le m.$$

Since (4.2) implies $\nabla u(x_0) = \nabla v(x_0)$, we have

$$\sum_{1 \le i,j \le m} (M_1^{ij} - M_2^{ij}) p_i p_j = \sum_{k,l=1}^{n} \eta_k \eta_l \frac{\partial^2 (u-v)^*}{\partial x_k \partial x_l}(x_*) \le 0,\ \forall p \in R^m,$$

where $\eta_k = \sum_{i=1}^m p_i a_{ik}(x_0)$, for $1 \le k \le n$. Hence $(M_1 - M_2)$ is negative semidefinite. Note also that $u(x_0) > v(x_0)$, $\nabla_h u(x_0) = \nabla_h v(x_0)$. Therefore the subellipticity and nonincreasing property of $\mathcal{F}$ implies

$$F(u(x_0), \nabla_h u(x_0), M_1) \le F(v(x_0), \nabla_h v(x_0), M_2). \tag{4.9}$$

This clearly contradicts with (4.7) and (4.8), since (4.7) implies

$$F(u(x_0), \nabla_h u(x_0), M_1) \ge 0,$$

and (4.8) implies

$$F(v(x_0), \nabla_h v(x_0), M_2) \le -c_\delta.$$

Therefore the theorem is proved.

Now we indicate the proof of claim 4.1. This claim follows from the compactness theorem (cf. [CIL]) among a family of viscosity sub/supersolutions to 2nd order PDEs. For simplicity, we only indicate how to prove (4.7). First we claim that u^ρ satisfies, in the sense of viscosity, in $B^E_{R_2\rho^{-1}}$,

$$\begin{aligned} &F(u(x_0+\rho x), \nabla_h u(x_0) + \rho X^\rho(x) u^\rho(x), \\ &\{\sum_{k,l=1}^{n} A^\rho_{ij,kl}(x) \frac{\partial^2 u^\rho}{\partial x_k \partial x_l}(x) + B^\rho_{ij,l}(x)(\frac{\partial u}{\partial x_l}(x_0) + \rho \frac{\partial u^\rho}{\partial x_l}(x))\}_{1 \le i,j \le m}) = 0, \end{aligned} \tag{4.10}$$

where $A^\rho_{ij,kl}(x) = a^\rho_{ik} a^\rho_{jl}(x)$, $B^\rho_{ij,l}(x) = \sum_{k=1}^n a^\rho_{ik} (\frac{\partial a_{jl}}{\partial x_k})^\rho$, $X^\rho(x) = (X_1^\rho(x), \cdots, X_m^\rho(x))$, $X_i^\rho(x) = X_i(x_0 + \rho x)$, $a^\rho_{ik}(x) = a_{ik}(x_0 + \rho x)$, and $(\frac{\partial a_{jl}}{\partial x_k})^\rho(x) = \frac{\partial a_{jl}}{\partial x_k}(x_0 + \rho x)$.

To see (4.10), let $(\bar{x}, \phi) \in B^E_{R_2\rho^{-1}} \times C^2(B^E_{R_2\rho^{-1}})$ be such that

$$0 = u^\rho(\bar{x}) - \phi(\bar{x}) \geq u^\rho(x) - \phi(x), \ \forall x \in B^E_{R_2\rho^{-1}}.$$

It is straightforward to see

$$\phi_\rho(x) \equiv u(x_0) + \langle \nabla u(x_0), x - x_0 \rangle_E + \rho^2 \phi(\frac{x - x_0}{\rho}), \forall x \in B^E_{R_2}(x_0)$$

satisfies

$$0 = u(x_0 + \rho\bar{x}) - \phi_\rho(x_0 + \rho\bar{x}) \geq u(x) - \phi_\rho(x), \ \forall x \in B^E_{R_2}(x_0).$$

This, combined with the fact that u is a viscosity subsolution to eqn.(1.?), implies

$$F(u(x_0 + \rho\bar{x}), \nabla_h \phi_\rho(x_0 + \rho\bar{x}), \nabla^2_h \phi_\rho(x_0 + \rho\bar{x})) \geq 0. \tag{4.11}$$

Direct calculations yield

$$\frac{\partial \phi_\rho}{\partial x_k}(x_0 + \rho\bar{x}) = \frac{\partial u}{\partial x_k}(x_0) + \rho \frac{\partial \phi}{\partial x_k}(\bar{x}), \ \forall 1 \leq k \leq n,$$

$$\frac{\partial^2 \phi_\rho}{\partial x_k \partial x_l}(x_0 + \rho\bar{x}) = \frac{\partial^2 \phi}{\partial x_k \partial x_l}(\bar{x}), \ \forall 1 \leq k, l \leq n.$$

Substituting these into (4.11), we obtain (4.10).

It is clear that, by taking $\rho \to 0$, (4.10) implies (4.7). This proves claim 4.1. ■

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